An Identification Algorithm That Is Insensitive to Initial Parameter Estimates

DALLAS G. DENERY*
NASA Ames Research Center, Moffett Field, Calif.

This paper is concerned with the estimation of parameters in a constant coefficient, linear system using measurements of the system input and output. Two general methods can be used to estimate these parameters: the equation error method and the output error method. The equation error method is characterized by a single step solution that does not require a prior estimate. Unbiased noise in the output, however, causes a bias in the estimated parameters. The output error method is characterized by iterative solution techniques that require a prior estimate of the unknown parameters. This method provides an unbiased estimate. In this paper, a single estimation procedure is presented that uses the best features of both methods. It does not require a prior estimate of the unknown parameters and unbiased noise in the output will not cause a bias in the final estimate. The method is applied to simulated and flight data.

Introduction

THIS paper is concerned with the estimation of a set of parameters that can be used to obtain a transfer function relationship between the input of a system and its output. Many parameter estimation methods for doing this can be classified into two categories, equation error methods[1-3] and output error methods.[3-5] An advantage of equation error methods is that the problem can often be made linear in terms of the unknown parameters, permitting a one-step algorithm. However, if there is unbiased noise in the measurements of the system output, the estimate will be biased.[6] Output error methods eliminate this bias,[4,7] but require iterative algorithms; a fairly accurate estimate of the unknown parameters is required to start such algorithms.

An equation error method can often be used to obtain an initial estimate of the unknown parameters and this estimate can then be used in an output error method. This has already been done successfully,[8] but two separate estimation procedures were used. Taylor,[9] on the other hand, incorporated a slight modification to an output error method and eliminated the necessity of using a separate procedure to obtain the initial estimates.

Taylor showed satisfactory results for one application where measurements of all output states were available. This paper extends his procedure to the multivariable case where there may be fewer measurements than state variables in the system model. This technique uses an equation error procedure, which is similar to a linear observer,[10] to obtain an initial estimate of the unknown parameters and then switches to a quasi-linearization[5] output error procedure. The mathematical structure of this equation error method is nearly identical to quasi-linearization. Because of this similarity, both procedures can be used in the same computational structure. This process will be referred to as the combined algorithm.

A feature of the present paper is the development of a canonical form for multioutput systems. When the unknown system is modeled in this canonical form, an identifiable set of the parameters is immediately defined and can be estimated by the combined algorithm. Although other canonical forms for multivariable systems are available,[11-13] the parameters in these forms are not located so that they can be estimated directly by the combined algorithm.

The combined algorithm is applied to the identification of the linearized longitudinal equations of motion for an aircraft using both simulated and flight data. Both single and multi-output examples are included.

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* Aerospace Engineer; also Ph.D. Candidate, Department of Applied Mechanics, Stanford University, Stanford, Calif.

Background

It is assumed that the structure of the differential equations of minimum order that can be used to relate the system input vector to the system output vector is known, and can be described by a set of first-order differential equations of the form

$$\dot{x} = Ax + Bu + e, x(0) = x_0, y = Cx + w \qquad (1)$$

where x is an $n \times 1$ state vector, u is a $p \times 1$ input vector, y is an $m \times 1$ output vector, e is zero mean process noise, and w is zero mean measurement noise. It is also assumed that the system input u is measured perfectly.

This set of first-order differential equations is not uniquely defined by the system's input and output. The input and output can also be described by any equations of the form

$$\dot{z} = Fz + Gu + v, z(0) = z_0, y = Hz + w \qquad (2)$$

where $z = Tx$, T is a nonsingular $n \times n$ matrix, and

$$F = TAT^{-1}, G = TB, H = CT^{-1}, v = Te \qquad (3)$$

In this paper we are only concerned with identifying some F, G, and H which can be used to represent the unknown system. Note F, G, and H contain $n(n + p + m)$ parameters.

Although all of the parameters in Eq. (2) cannot be uniquely determined by the input and output, if some of the parameters are specified, then the remaining parameters might be uniquely determined as a function of the specified parameters. There are several canonical forms for F, G, and H that contain a maximum of $n(m + p)$ free parameters; the other n^2 parameters being specified.[11-13] However, the parameters in these canonical forms are not located so that they can be identified directly by using the combined algorithm. In this paper a canonical form for F, G, and H is constructed that contains a maximum of nm free parameters in the F and H matrices and np parameters in the G matrix. These parameters can be estimated using the combined algorithm.

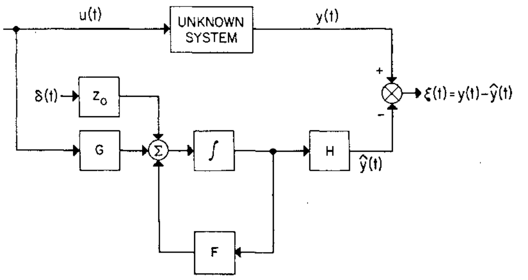

Fig. 1 Output error structure.

Example I

To illustrate the aforementioned idea, consider the single input, single output, second-order system given by

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}f_{11} & f_{12}\\ f_{21} & f_{22}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}g_{11}\\ g_{21}\end{bmatrix}[u] \quad \begin{bmatrix}z_1(0)\\ z_2(0)\end{bmatrix} = 0 \tag{4a}$$

$$y = [h_{11} \quad h_{12}]\begin{bmatrix}z_1\\ z_2\end{bmatrix} \tag{4b}$$

The Laplace transform of Eq. (4) is given by

$$y(s)/u(s) = (d_1 s + d_0)/(s^2 + c_1 s + c_0) \tag{5}$$

where

$$d_1 = g_{11}h_{11} + g_{21}h_{12}$$
$$d_0 = g_{11}[-h_{11}f_{22} + h_{12}f_{21}] + g_{21}[-h_{12}f_{11} + h_{11}f_{12}]$$
$$c_1 = [-f_{11} - f_{22}],\ c_0 = [f_{22}f_{11} - f_{12}f_{21}]$$

Since the response of Eq. (4) is completely specified by the four coefficients d_1, d_0, c_1, and c_0, the eight parameters in F, G, and H cannot be uniquely defined. However, four of the unknown parameters in F, G, and H can be determined in terms of the other four parameters. By choosing these other parameters we are, in essence, constraining the structure of F, G, and H so that the four remaining parameters are uniquely defined by the input and output relationships. One way of constraining the structure of F, G, and H is to set $h_{11} = 1$, $h_{12} = 0$, $f_{12} = 1$, and $f_{22} = 0$. The four remaining parameters in F, G, and H are then uniquely defined by the relationships $d_1 = g_{11}$, $d_0 = g_{21}$, $c_1 = -f_{11}$, $c_0 = -f_{21}$. This particular choice of F, G, and H is a well-known canonical form for single output systems, the parameters of which can be estimated using the combined algorithm. This type of canonical form will be presented for the multioutput system in this paper.

Estimation Technique

A common procedure for estimating the unknown parameters in Eq. (2) is to build a model of the system,

$$\dot{\hat{z}} = F\hat{z} + Gu,\ \hat{z}(0) = z_0,\ \hat{y} = H\hat{z} \tag{6}$$

and adjust the unknown parameters of the model to minimize the function

$$J = \tfrac{1}{2}\int_0^{t_f} (y - \hat{y})^T W(y - \hat{y})dt \tag{7}$$

where W is a weighting matrix used to express the relative confidence in the measurements. This is illustrated in Fig. 1. The difficulty with this method is that the response of the model $\hat{y}$ is a nonlinear function of the unknown parameters in F and H and therefore J must be minimized by an iterative procedure.[5,14] One method for solving this problem is included in the combined algorithm and is presented later in this section. However, if no noise is present in the system, this problem can be converted to a linear problem. This linear formulation constitutes the equation error portion of the combined algorithm. The effect of applying the linear formulation of the problem to the identification of a system with noise will be briefly discussed.

Equation Error Procedure

The linear formulation is based on the idea that, for a perfect model and in the absence of noise, the output of Eq. (6) will equal the measured output exactly and therefore the difference $y - \hat{y}$ equals zero. Under these conditions, this difference can be fed back to the model through arbitrary gains K and L without changing the model response $\hat{y}$. The equations for the model with this error feedback are

$$\dot{\hat{z}} = F\hat{z} + Gu + K[y - H\hat{z}],\ \hat{z}(0) = z_0 \tag{8a}$$

$$\hat{y} = H\hat{z} + L[y - H\hat{z}] \tag{8b}$$

which by combining terms can be rewritten

$$\dot{\hat{z}} = (F - KH)\hat{z} + Gu + Ky,\ \hat{z}(0) = z_0 \tag{9a}$$

$$\hat{y} = (I - L)H\hat{z} + Ly \tag{9b}$$

The first equation in Eqs. (8) and (9) is the state observer equation as studied by Luenberger.[10] It is shown in the next section that the parameters of $F - KH$ and $(I - L)H$ can be chosen independently of the unknown system parameters by using a maximum of nm parameters in K and L, provided the structure of the system is known. These choices will be denoted by F_N and H_N, respectively. It is convenient to define $\delta G = G - G_N$ and $\delta z_0 = z_0 - z_{N0}$ where G_N and z_{N0} can be interpreted as initial estimates of G and z_0 and can include any known parameters. If these definitions are used in Eq. (9), $\hat{y}$ can be written

$$\dot{\hat{z}} = F_N\hat{z} + G_N u + \delta G u + Ky,\ \hat{z}(0) = z_{N0} + \delta z_0,\ \hat{y} = H_N\hat{z} + Ly \tag{10}$$

The advantage of using this formulation to model the unknown system is that since F_N, G_N, H_N, and z_{N0} can be chosen, the unknown parameters are contained in K, L, δG, and δz_0. These parameters are coefficients of known forcing functions and therefore affect the model response $\hat{y}$ linearly.

Because of the linearity, $\hat{y}$ can be expressed

$$\hat{y} = y_N + f(t)\cdot\gamma \tag{11}$$

where y_N is the response of the equations

$$\dot{z}_N = F_N z_N + G_N u,\ z_N(0) = z_{N0},\ y_N = H_N z_N \tag{12}$$

γ is a vector containing the parameters in K, δG, L, and δz_0, and $f(t)$ is the gradient of $\hat{y}$ with respect to these parameters {i.e., $f(t) = [\hat{y}_{\gamma 1}, \hat{y}_{\gamma 2}, \ldots]$}. When the expression for $\hat{y}$ given in Eq. (11) is substituted into Eq. (7), J becomes quadratic in the unknown parameters. The estimates can be obtained by differentiating J with respect to the unknown parameters, setting the resulting equations equal to zero, and solving for the estimate of $\gamma, \hat{\gamma}$. This procedure results in the well-known solution,[5,7,14]

$$\hat{\gamma} = \left[\int_0^{t_f} f^T(t)Wf(t)dt\right]^{-1}\left[\int_0^{t_f} f^T(t)W(y - y_N)dt\right] \tag{13}$$

The columns of $f(t)$ are the numerical solutions of the differential equations

$$\dot{\hat{z}}_{\gamma i} = F_N\hat{z}_{\gamma i} + (\partial K/\partial\gamma_i)y + [\partial(\delta G)/\partial\gamma_i]u \tag{14a}$$

$$\hat{z}_{\gamma i}(0) = \partial(\delta z_0)/\partial\gamma_i,\ \hat{y}\gamma_i = H_N\hat{z}\gamma_i + (\partial L/\partial\gamma_i)y \tag{14b}$$

where the partial derivatives of K, δG, L, and δz_0 with respect to γ_i are all equal to zero except for a 1 in the location of the specific parameter γ_i. Equations (14) will be referred to as the sensitivity equations. Certain simplifications in the computation of these sensitivities can be accomplished by using a procedure similar to that discussed in Ref. 15.

The estimates for F, G, H, and z_0 are determined from the estimates of K, δG, L, and δz_0 by the relationships

$$\hat{H} = (I - \hat{L})^{-1}H_N,\ \hat{G} = G_N + \widehat{\delta G} \tag{15a}$$

$$\hat{F} = F_N + \hat{K}\hat{H},\ \hat{z}_0 = z_{N0} + \widehat{\delta z_0} \tag{15b}$$

In this way the identification problem has been reduced to a single operation involving the numerical solutions of Eq. (12–15).

Example II

In Example I, it was shown that a single output second-order system could be realized by the equations

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}f_{11} & 1\\ f_{21} & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}g_{11}\\ g_{21}\end{bmatrix}[u],\ z_0 = 0$$

$$y = [1 \quad 0]\begin{bmatrix}z_1\\ z_2\end{bmatrix}$$

Consider the identification of the parameters in F, G, and z_0.

If the equations are in this form, then

$$F_N = \begin{bmatrix}f_{11}{}^N & 1\\ f_{21}{}^N & 0\end{bmatrix},\ \text{and } H_N = [1 \quad 0]$$

can be chosen in Eq. (10) independently of f_{11} and f_{21} and still be related to F and H by the relationships $F_N = F - KH$ and $H_N = (I - L)H$ by setting

$$K = \begin{bmatrix}k_{11}\\ k_{21}\end{bmatrix} = \begin{bmatrix}f_{11} - f_{11}{}^N\\ f_{21} - f_{21}{}^N\end{bmatrix},\ L = 0$$

Since the parameters in G and z_0 are not known, G_N and z_{N0} can be chosen arbitrarily. An identification of F, G, and H can be obtained by using Eqs. (12–15) where the unknown parameters in γ are the elements in the vectors K, δG, and δz_0. Note that the identification of the parameters in F depends on the fact that these parameters are coefficients of a measured state, $z_1 = y$. It is always possible to describe a single output system this way. However, this is not generally true for a multioutput system and the matrix L must be included in the analysis. This will become evident in the section on canonical forms.

It has been assumed that the system is noise free. If there is noise in the system, the output of Eq. (10) will not equal the output of Eq. (6) and therefore cannot correctly be used to replace Eq. (6). Nevertheless, if Eq. (10) is used in the expression for J, the procedure can be carried out and an estimate for the parameters obtained. However, these estimates will be biased. This means that the expected value of the error in the parameter estimates will not be zero. It can be shown that the size of the bias is equal to a constant plus a term proportional to the size of the estimates $\hat{K}$ and $\hat{L}$. A proof for the existence of this type of bias can be found in Ref. 6. If the initial estimates of K and L are extremely large, the bias error can usually be reduced by choosing a new F_N and H_N equal to the estimates for F and H and repeating the process. Because of the constant in the bias term, the bias usually cannot be completely eliminated by this process. This is illustrated in the application section.

Output Error Procedure

The main reason for using an output error procedure is that the presence of unbiased noise in the system does not cause a bias in the parameter estimates. One well-known procedure for minimizing Eq. (7) subject to the differential equations, Eqs. (6), is the method of quasi-linearization.[5] This procedure yields an unbiased estimate of the parameters if initiated from a sufficiently accurate prior estimate of these parameters.[4,7] Generally the biased estimates obtained from the equation error procedure just described are accurate enough (for reasonable amounts of noise) to provide a good initial estimate to use in the method of quasi-linearization.

In this section, it is shown that if F_N, G_N, H_N, and z_{N0} are interpreted as initial estimates of the unknown system and y_N from Eq. (12) is substituted for y in Eq. (10), then the equation error procedure developed in the first part of this section becomes one step in an iterative procedure that is nearly identical to the quasi-linearization procedure. This similarity of structure is the basis of the combined algorithm.

The basic idea in quasi-linearization is that $\hat{y}$, the response of Eq. (6), can be approximated by a trajectory y_N that is based on the best initial estimates of the unknown parameters plus a linearized correction about this estimate. If the initial estimates of F, G, H, and z_0 are defined as F_N, G_N, H_N, and z_{N0}, respectively, then $\hat{y}$ can be approximated by $\hat{y}_A$ where

$$\dot{\hat{z}}_A = F_N\hat{z}_A + [F - F_N]z_N + [G_N + \delta G]u,\ \hat{z}_{A0} = z_{N0} + \delta z_0 \tag{16a}$$

$$\hat{y}_A = H_N\hat{z}_A + [H - H_N]z_N \tag{16b}$$

and z_N is the response of

$$\dot{z}_N = F_N z_N + G_N u,\ z_N(0) = z_{N0},\ y_N = H_N z_N \tag{17}$$

If the same constraints are used on the structure of F, G, and H as were applied in the equation error formulation, $F - F_N$ and $H - H_N$ can be replaced by KH and LH, respectively. Since in the output error procedure $F - F_N$ and $H - H_N$ are considered small, K and L can be considered small; hence,

$$KH = K[H_N + LH] \approx KH_N \tag{18a}$$

$$LH + L[H_N + LH] \approx LH_N \tag{18b}$$

Making these substitutions in Eqs. (16) we obtain

$$\dot{\hat{z}}_A = F_N\hat{z}_A + G_N u + \delta G u + Ky_N \quad z_{A0} = z_{N0} + \delta z_0 \tag{19a}$$

$$\hat{y}_A = H_N\hat{z}_A + Ly_N \tag{19b}$$

which are identical to Eq. (10) except that y_N has replaced y. If $\hat{y}_A$ is used in Eq. (7) in place of $\hat{y}$, the minimization problem is reduced to the minimization of a quadratic form identical to that given in the equation error procedure. The solution is given by Eqs. (12–15), with the exception that y_N is used in place of y in Eq. (14). A new estimate of the unknown parameters is obtained and an iterative procedure is established for minimizing Eq. (7).

Combined Algorithm

The idea behind the combined algorithm is now evident. The equation error procedure and output error procedure are identical except for the computation of $f(t)$. The only difference here is whether measured or estimated data are used in the sensitivity equations, Eqs. (14). If the measured data are used, the procedure provides an estimate of F, G, H, and z_0 in a single operation essentially independent of the initial choice of F_N, G_N, H_N, and z_{N0}. In the absence of noise, this estimate is the same as the quasi-linearization estimate; but if there is noise in the system, this estimate will be biased. Choosing a new F_N, G_N, H_N, and z_{N0} equal to the estimates of F, G, H, and z_0 and repeating the equation error process usually reduces the bias error. On achieving the best estimate by the equation error procedure, the combined algorithm switches y_N for y in the sensitivity equations, Eqs. (14). This implements the output error procedure which generally converges to the unbiased estimate very rapidly.

The combined algorithm is indicated diagrammatically in Fig. 2. The switch in the upper center of the diagram represents the only change required to switch from the equation error formulation to the output error formulation. For the initial set of iterations, the switch is to the right. After that, it is to the left. The rest of the computational structure remains unchanged. The components of $f(t)$ are computed by the numerical solution of the sensitivity equations, Eqs. (14), which are labeled in the figure. The outputs of the sensitivity equations are used to form the products $f(t)^TWf(t)$ and $f(t)^TW(y - y_N)$ which are integrated simultaneously in order to reduce storage requirements. The estimate γ is obtained by the solution of Eq. (13) at the final time t_f, and the unknown parameters are computed using Eq. (15). The process is then repeated as indicated where the superscript (1) indi-

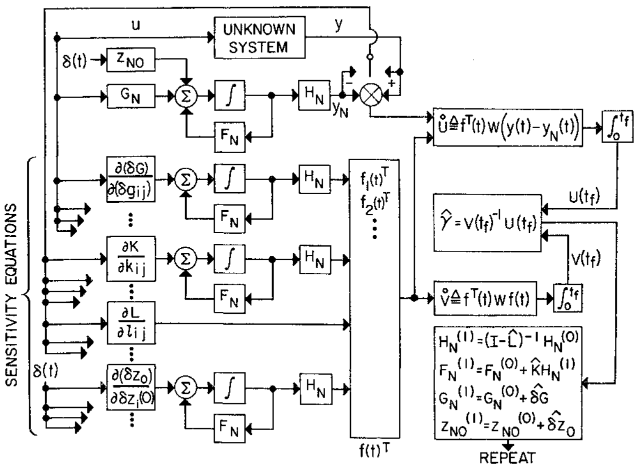

Fig. 2 Combined algorithm.

cates the new estimate and the superscript (0) represents the previous estimate.

A Canonical Form for Multioutput Systems

It has been asserted that the parameters of $F - KH$ and $(I - L)H$ can be chosen independently of the numerical values of the unknown parameters by using a maximum of nm parameters in K and L. If the system is described by a set of equations in which some of the states are measured directly and the unknown parameters in F are coefficients of measured states, the F_N and H_N can be chosen equal to F and H except for the unknown parameters in F. L can be set equal to zero and the parameters of K are equal to the differences between the unknown parameters in F and the values chosen for these parameters in F_N. The case in which all the states are measured is an example of this type.

Unfortunately, unless all the states are measured, the system equations will usually have some unknown parameters that are coefficients of states that are not measured. It was shown in Example II that all single output systems can be rewritten with the unknown parameters as coefficients of the measured state. In this section a canonical form for multi-output systems is developed which is analogous to one developed by Luenberger[11] for multi-input systems. The single output canonical form used in Example II is included as a special case. In general, this canonical form will contain some unknown parameters in H as well as in F. Nevertheless, F_N and H_N can be chosen equal to F and H except for the unknown parameters and used with the combined algorithm.

In order to write the canonical form, it is only necessary to determine the first n linearly independent rows in the observability matrix,

$$O_b = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} \tag{20}$$

for the system Eq. (1). In essence, this amounts to knowing the structure of the unknown system, but not necessarily the numeric values of the parameters in this structure. Define p_i to be the number of rows in this linearly independent set that involves a multiplication by the ith row of C. The p_i completely determine the structure of the canonical form. The transformation T_c, which can be used to transform the system Eq. (1) to its canonical form, is constructed in the Appendix.

If the p_i for the system are known, the canonical form is given by

$$\dot{z} = Fz + Gu,\ z(0) = z_0,\ y = Hz \tag{21}$$

where F and H are given in Fig. 3 and where G contains all x's. The empty portions of F and H represent zeros and the symbol I is the identity matrix.

If the system is modeled by Eq. (21), using Fig. 3 for F and H, the free parameters denoted by the x's, would still not be identifiable. By tedious inspection we have shown that some additional parameters F and H can be set to zero by the relationships

$$\text{if } p_i < p_j - k \text{ then } f_{s_j+k,s_i} = 0 \qquad k = 0, \ldots p_j - p_i - 1 \tag{22}$$

$$\text{if } p_i \le p_j\ i \ne j \text{ then } h_{j,s_i} = 0 \tag{23}$$

where $f_{i,j}$ and $h_{i,j}$ are elements in F and H, respectively, and the subscript s_i is defined by

$$s_i = 1 + \sum_{j=1}^{i-1} p_j$$

The remaining undefined parameters in F, G, and H are uniquely defined by the system input and output relationships. There is a maximum of nm of these parameters in F and H and np of these parameters in G.

In many applications (all that we have considered), it is possible to order the measurements y so that the first n rows of O_b, Eq. (20), are linearly independent. The canonical form for this case is examined in detail because of its frequency of occurrence. If r and q are defined as the remainder and quotient of n/m respectively, then $p_i = q + 1$ for $i \le r$ and $p_i = q$ for $i > r$ and the canonical form for F and H is given in Fig. 4. The columns and rows of F are divided into m groupings. The x's in the first rows of the first r row groupings and in the last m-r column groupings are equal to zero. The x's in H are all equal to zero except for those occurring in the last $(m$-$r)$ rows of the first r column groupings.

If the system is modeled in its canonical form, then a F_N and H_N can be chosen in the same canonical form with arbitrary values inserted for the x's and still be related to F and H by the equalities $F_N = F - KH$, $H_N = (I - L)H$. This completes the original assertion that was stated at the beginning of this section. The minimum number of parameters in K and L required to establish these equalities constitute an identifiable set of parameters.

Example III

Let us consider a third-order system with two outputs described by the equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 1 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} [u]$$
$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{24}$$

If x_1, x_2, and x_3 are interpreted as perturbations in attitude rate, filtered vertical acceleration, and angle of attack, Eq.

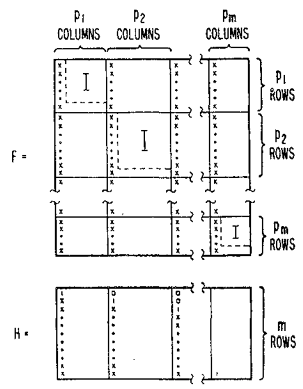

Fig. 3 General canonical structure.

(24) describes the short-period equations of motion for longitudinal flight [see Eq. (29)].

Since all the unknown parameters in Eq. (24) are not coefficients of the measured state, the combined algorithm cannot be used to identify the parameters directly. We will therefore transform these equations to their canonical form. The observability matrix for the system is given by

$$O_b = \begin{bmatrix} c_{(1)} \\ c_{(2)} \\ c_{(1)}A \\ c_{(2)}A \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ & \cdot & \\ & \cdot & \end{bmatrix} \tag{25}$$

The first three rows of the observability matrix are linearly independent, provided a_{13} is not equal to zero. If we interpret Eqs. (24) as the short-period equations of motion for an airplane, the parameter a_{13} is equal to $(M_{\dot{\alpha}}Z_\alpha/I_y m u_0) + (M_\alpha/I_y)$, which although unknown is not equal to zero. Therefore the values of p_i for the system are $p_1 = 2$ and $p_2 = 1$. On the basis of Fig. 4, the canonical form for these equations is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} f_{11} & 1 & 0 \\ f_{21} & 0 & f_{23} \\ f_{31} & 0 & f_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ h_{21} & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{26}$$

The matrices $F_N = F - KH$ and $H_N = (I - L)H$ can be chosen

$$F_N = \begin{bmatrix} f_{11}^N & 1 & 0 \\ f_{21}^N & 0 & f_{23}^N \\ f_{31}^N & 0 & f_{33}^N \end{bmatrix}, \text{ and } H_N = \begin{bmatrix} 1 & 0 & 0 \\ h_{21}^N & 0 & 1 \end{bmatrix} \tag{27}$$

where

$$K = \begin{bmatrix} k_{11} & 0 \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{bmatrix}, \text{ and } L = \begin{bmatrix} 0 & 0 \\ l_{21} & 0 \end{bmatrix} \tag{28}$$

Estimates for F, G, and H can be obtained by including the parameters k_{11}, k_{21}, k_{31}, k_{22}, k_{32}, l_{21}, δg_{11}, δg_{21}, and δg_{31} in the vector γ and using Eqs. (27) and (28) in the combined algorithm.

If the equations are actually transformed by means of the transformation T_c given in the Appendix, it can be shown that f_{23} is equal to zero. Therefore, the parameter k_{22} could be eliminated from the estimation procedure by choosing $f_{23}^N = 0$.

Application

Application of Combined Algorithm to Simulated Data

The short-period dynamics of the C-8 airplane in the landing approach were simulated and the attitude rate response due to an elevator deflection was computed. The initial conditions were set equal to zero. Three different noise sequences having a variance equal to 0.003 rad squared and a 0.2-sec correlation time constant were added to this attitude rate data to give three different runs. These same three noise sequences were also subtracted from the attitude rate data to give three additional runs making a total of six.

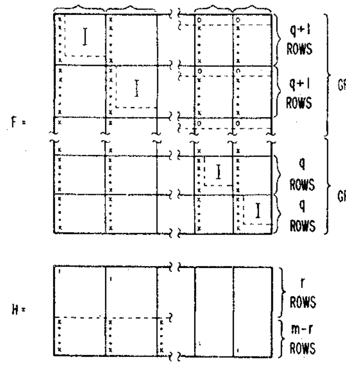

Fig. 4 Special case canonical structure.

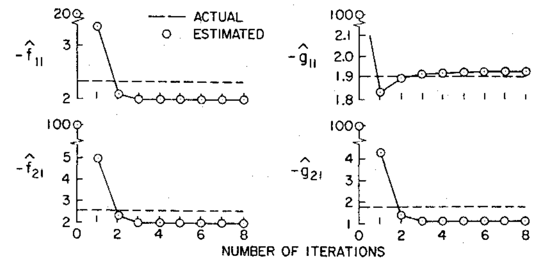

Fig. 5 Application of equation error method illustrating convergence and bias.

For this example it is convenient to model the unknown system in the canonical form discussed in Example II. z_1 can be interpreted as the attitude rate, z_2 as a linear combination of attitude rate and angle of attack, and u as the elevator deflection.

In order to illustrate the characteristics of the equation error and output error portions of the combined algorithm independently, the six runs were first analyzed by means of the equation error portion of the algorithm. The estimated parameters of the canonical form were averaged over the six runs to reduce the error in these estimates due to variance and thereby illustrate the bias error. These averaged estimates are plotted in Fig. 5 against the number of iterations. The initial choice of the parameters, F_N, H_N, and G_N, denoted by the zero estimate, was purposely made considerably different from the actual values to emphasize the insensitivity of the convergence on this initial estimate. By the second iteration, the procedure has essentially reached a steady-state value for the unknown parameters and subsequent iterations do not significantly change these estimates. The important point to note is that there is a very definite bias in these answers.

To illustrate that the bias observed in Fig. 5 can be eliminated by switching to the output error formulation, the final averaged estimates obtained by the equation error formulation in Fig. 5 were used to initiate the output error formulation for the same six runs. The average values of these estimates are plotted against the number of iterations in Fig. 6. As is shown, the bias is quickly removed. The final averaged parameter estimates are very close to the actual values.

Application of Combined Algorithm to Flight Data

Measurements of the attitude rate and elevator deflection for the C-8 airplane in the landing approach configuration were used to identify the coefficients of the transfer function relating pitch rate to elevator deflection. The aircraft was excited by a doublet type elevator deflection and the data were measured over a 4-sec interval. Because of the type of

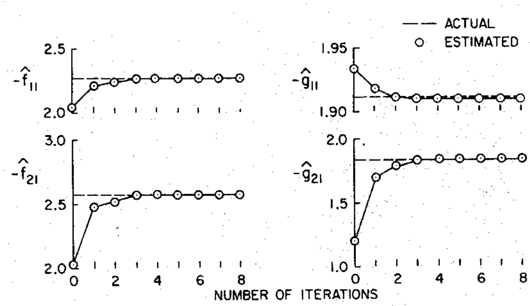

Fig. 6 Application of output error method illustrating elimination of bias.

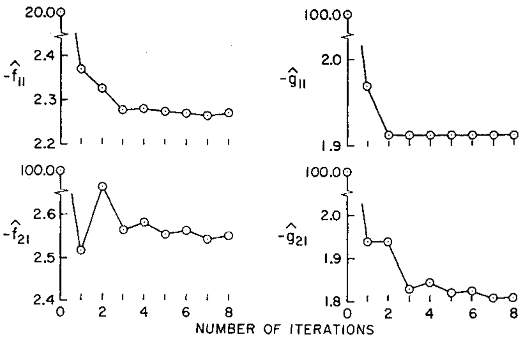

Fig. 7 Application of combined algorithm to flight test data short period dynamics.

input and the short duration of data, the phugoid (long-period) mode was not noticeably excited. For this reason, the system was again represented by the short-period dynamics and was modeled by the single-output canonical form used in Example II. The estimated coefficients are plotted against the number of iterations in Fig. 7. The initial or zero estimate was purposely made considerably different from the expected system parameters to again emphasize the insensitivity of the method on this initial estimate. After four iterations, the parameters settled to a steady-state value. The equation error formulation was used during the first two iterations. The combined algorithm then switched to the output error formulation.

The accuracy of this identification is indicated in Fig. 8. The time history of the elevator input is shown in radians on the left side of the figure. This input was used together with the identified system dynamics to compute an estimated attitude rate. The computed attitude rate is shown by the solid line on the right side of the figure and the measured attitude rate by the symbols. Clearly, the estimated transfer function provides a very good relationship between the input and output data.

In order to illustrate the combined algorithm applied to a higher-order system, measurements of the attitude rate, forward velocity, vertical acceleration, angle of attack, and elevator deflection for the C-8 airplane in the landing approach configuration were made and used to identify the parameters in the complete linearized longitudinal equations of motion. In this case, there were approximately 17 sec of data and both the short- and long-period modes were excited. The vehicle was modeled by the equations

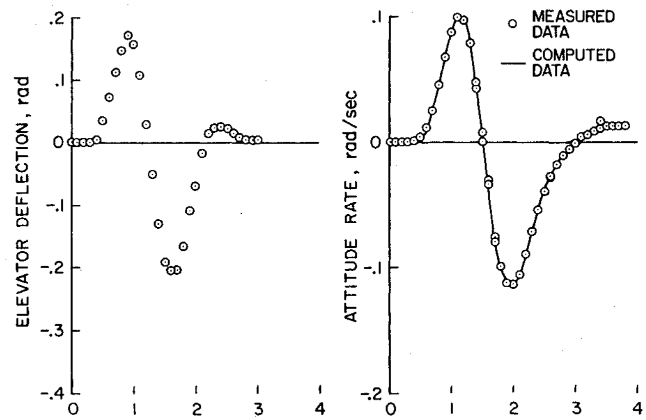

Fig. 8 Comparison of measured and estimated data.

cients of measured states. The dependency between the parameters in the fourth and fifth rows of Eq. (29) was maintained in the identification (i.e., $f_{41} = -20u_0f_{51}$, $f_{45} = -20u_0f_{55}$, and $g_{41} = -20u_0g_{51}$).

Since this is a multioutput situation, an appropriate weighting matrix W must be chosen for use in Eq. (7). For this example, the weightings on u, q, a_z, and α were chosen to be $1/(1\text{f ps})^2$, $1/(1°/\text{sec})^2$, $1/(1\ \text{ft/sec}^2)^2$, and $1/(2°)^2$, respectively.

The results of this identification are shown in the 10 columns of Table 1. The parameter symbols are given in the first column. The initial estimates used to start the algorithm are given in the second column. The third and fourth columns give the estimates after the first two iterations using the equation error formulation. The remaining columns correspond to successive iterations using the output error procedure. Significant changes in the unknown parameters do not occur after the third or fourth iteration.

The identified parameters were used with the measured input to compute time histories of the velocity, attitude rate, vertical acceleration, and angle-of-attack perturbations. The computed and measured quantities are compared in Fig. 9. As in the case of the first example, the estimated parameters provide a very good relationship between the input and output data.

Concluding Remarks

A method of parameter estimation has been presented that combines the best properties of the equation error and output

$$\begin{bmatrix} \dot{u} \\ \dot{\theta} \\ \dot{q} \\ \dot{a}_z \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} \dfrac{x_u}{m} & -g & -w_0 & 0 & \dfrac{x_\alpha}{m} \\ 0 & 0 & 1 & 0 & 0 \\ \dfrac{M_u}{I_y} + \dfrac{M_{\dot{\alpha}}z_u}{I_y m u_0} & 0 & \dfrac{M_{\dot{\alpha}} + M_q}{I_y} & 0 & \dfrac{M_{\dot{\alpha}}z_\alpha}{I_y m u_0} + \dfrac{M_\alpha}{I_y} \\ -20\,\dfrac{z_u}{m} & 0 & 0 & -20 & -20\,\dfrac{z_\alpha}{m} \\ \dfrac{z_u}{m u_0} & 0 & 1 & 0 & \dfrac{z_u}{m u_0} \end{bmatrix} \begin{bmatrix} u \\ \theta \\ q \\ a_z \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{M_{\dot{\alpha}}z\delta_e}{I_y m u_0} + \dfrac{M\delta_e}{I_y} \\ -20\,\dfrac{z\delta_e}{m} \\ \dfrac{z\delta_e}{m u_0} \end{bmatrix} [\delta_e] \quad (29)$$

The states u, θ, q, and α are the perturbations in forward velocity, attitude, attitude rate, and angle of attack from level steady-state flight; a_z is a filtered measurement of the vertical acceleration. The filter time constant was 0.05 sec, and this is indicated by the factor of 20.0 occurring in the equation for acceleration. The control variable δ_e is the elevator deflection. The trim velocity u_0 and the gravitational constant g are assumed known. The other parameters in the F and G matrices depend on the aerodynamic and mass characteristics of the vehicle and are considered unknown.

In this example, it is not necessary to go to the canonical form because the unknown parameters in Eq. (29) are coeffi-

error techniques. In the absence of noise, the procedure provides a weighted least squares estimate for the unknown parameters in a single operation. If there is noise in the system, this initial estimate will be biased. The bias error can be removed by applying the procedure iteratively.

A canonical form for multioutput systems is presented which can be estimated using the combined algorithm. Modeling the system in its canonical form provides a set of identifiable parameters.

The combined algorithm has been applied successfully to the identification of the parameters in the longitudinal equations of aircraft motion using both simulated and flight data.

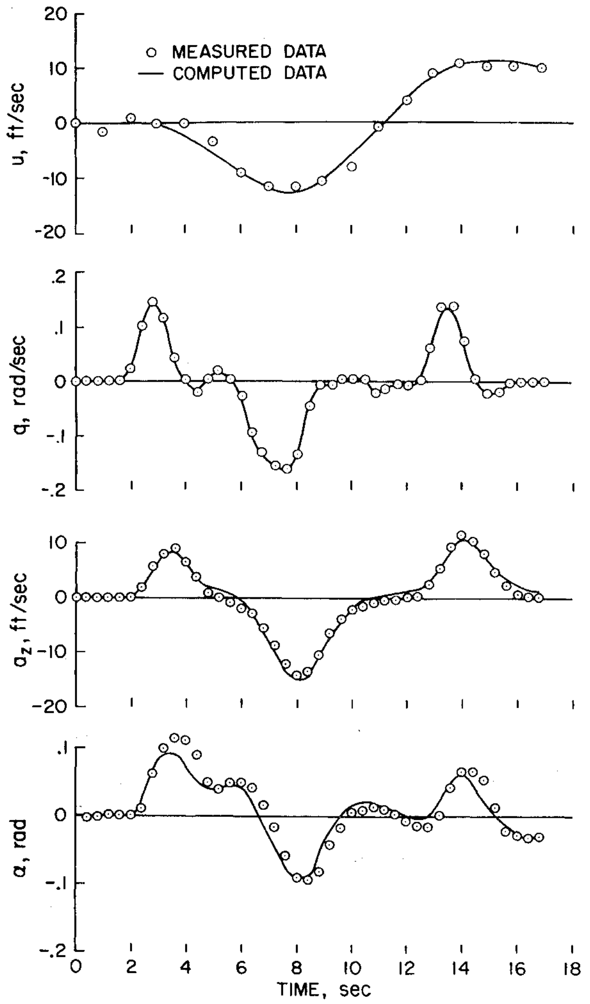

Fig. 9 Comparison of measured and estimated data.

Appendix: Canonical Transformation

It was stated in the section titled "A Canonical Form for Multioutput Systems" that any observable system could be transformed to the canonical form given in Fig. 3 and by Eqs. (21–23) by means of a nonsingular transformation matrix T_c. In this Appendix, the appropriate transformation T_c is constructed.

Arrange the first n linearly independent rows of the observability matrix (20) to form a nonsingular matrix P,

$$P = \begin{bmatrix} c_{(1)} \\ c_{(1)}A \\ \cdot \\ \cdot \\ c_{(1)}A^{p_1-1} \\ c_{(2)} \\ \cdot \\ \cdot \\ c_{(m)}A^{p_m-1} \end{bmatrix}$$

where $c_{(i)}$ is the ith row of C and where p_i is the number of rows in this linearly independent set involving a multiplication by the ith row of C. Define $q^{(j)}$ as the jth column of $P^{-1} \triangleq Q$. The inverse of the canonical transformation matrix then can be constructed,

$$T_c^{-1} = [A^{(p_1-1)}q^{(l_1)}, \ldots, q^{(l_1)}, \ldots, A^{(p_m-1)}q^{(l_m)}, \ldots, q^{(l_m)}]$$

Table 1 Application of combined algorithm to multioutput data

PARAMETER AND INITIAL ESTIMATES		COMBINED ESTIMATION ALGORITHM							
SYMBOL	INITIAL ESTIMATES	EQUATION ERROR		OUTPUT ERROR					
$\frac{X_u}{m}$	-0.01	-0.017	-0.020	-0.020	-0.020	-0.020	-0.020	-0.020	-0.020
$\frac{M_u}{I_{yy}} + \frac{M_{\dot\alpha} Z_u}{I_{yy} m u_0}$	0.0	.002	.002	.003	.003	.003	.003	.003	.003
$\frac{Z_u}{m u_0}$	0.0	-.003	-.003	-.004	-.004	-.004	-.004	-.004	-.004
$\frac{M_{\dot\alpha} + M_q}{I_{yy}}$	-3.0	-1.373	-1.308	-1.602	-1.58	-1.584	-1.585	-1.587	-1.588
$\frac{X_\alpha}{m}$	0.0	27.837	33.144	33.887	33.685	33.705	33.720	33.731	33.736
$\frac{M_{\dot\alpha} Z_\alpha}{I_{yy} m u_0} + \frac{M_\alpha}{I_{yy}}$	0.0	-.404	-.578	-.541	-.567	-.564	-.564	-.563	-.562
$\frac{Z_\alpha}{m u_0}$	-1.0	-.83	-.808	-.739	-.736	-.737	-.737	-.737	-.737
$\frac{M_{\dot\alpha} Z_{\delta_e}}{I_{yy} m u_0} + \frac{M_{\delta_e}}{I_{yy}}$	-1.0	-1.466	-1.441	-1.651	-1.654	-1.656	-1.656	-1.657	-1.658
$\frac{Z_{\delta_e}}{m u_0}$	-1.0	-.007	.012	.005	.006	.005	.005	.005	.005

where l_i is defined by

$$l_i = \sum_{j=1}^{i} p_j$$

References

[1] Greenberg, H., "A Survey of Methods for Determining Stability Parameters of an Airplane from Dynamic Flight Measurements," TN 2340, April 1951, NACA.

[2] Shinbrot, M., "On the Analysis of Linear and Nonlinear Dynamical Systems from Transient-Response Data," TN 3288, Dec. 1954, NACA.

[3] Lessing, H. C. and Crane, D. F., "The Use of Integral Transforms in the Estimation of Time Variable Parameters," TN D-5194, May 1969, NASA.

[4] Elkind, J. I., "Further Studies of Multiple Regression Analysis of Human Pilot Dynamic Response; A Comparison of Analysis Techniques and Evaluation of Time-Varying Measurements," ASD-TDR-63-618, Rept. 1069, March 1964, Bolt, Beranek and Newman Inc., Cambridge, Mass.

[5] Bellman, R., "On the Construction of a Mathematical Theory of the Identification of Systems," RM-4769-PR, Nov. 1966, Rand Corp., Santa Monica, Calif.

[6] Young, P. C., "Process Parameter Estimation and Adaptive Control," *Theory of Self-Adaptive Control Systems*, edited by P. Hammond, Plenum, New York, 1966, pp. 118–139.

[7] Schalow, R. D., "Quasilinearization and Parameter Estimation Accuracy," Ph.D. dissertation, 1967, Dept. of Electrical Engineering, Syracuse Univ., Syracuse, N.Y.

[8] Larson, D. B. and Fleck, J. T., "Identification of Parameters by the Method of Quasilinearization," Rept. 164, May 1968, Cornell Aeronautical Lab., Buffalo, N.Y.

[9] Taylor, L. W., Jr., Iliff, K. W., and Powers, B. G., "A Comparison of Newton-Raphson and Other Methods for Determining Stability Derivatives from Flight Data," AIAA Paper 69-315, Houston, Texas, 1969.

[10] Luenberger, D. G., "Observing the State of a Linear System," *IEEE Transactions on Military Electronics*, Vol. MIL-8, No. 2, April 1964, pp. 74–80.

[11] Luenberger, D. G., "Canonical Forms for Linear Multivariable Systems," *IEEE Transactions on Automatic Control*, Vol. AC-12, No. 3, June 1967, pp. 290–293.

[12] Widnall, W. S., "Applications of Optimal Control Theory to Computer Controller Design," Research Monograph 48, The MIT Press, Cambridge, Mass., 1968, pp. 47–56.

[13] Vogt, W. G. and Gunderson, R. W., "Canonical Realizations," *Proceedings of the First Asilomar Conference on Circuits and Systems*, co-sponsored by Electrical Engineering Dept., Naval Postgraduate School, Monterey and Santa Clara Univ., Vol. 1, 1967, pp. 782–790.

[14] Bryson, A. E., Jr. and Ho, Y. C., "Applied Optimal Control," Blaisdell, Waltham, Mass., 1969, pp. 229–236.

[15] Wilkie, D. F. and Perkins, W. R., "Generation of Sensitivity Functions for Linear Systems Using Low-Order Models," *IEEE Transactions on Automatic Control*, Vol. AC-14, No. 2, April 1969, pp. 123–130.